The Moyal bracket and the dispersionless limit of the KP hierarchy

1995 J. Phys. A: Math. Gen. 28 1967

(http://iopscience.iop.org/0305-4470/28/7/018)

J. Phys. A: Math. Gen. 28 (1995) 1967–1975. Printed in the UK

The Moyal bracket and the dispersionless limit of the KP hierarchy

I A B Strachan†

Department of Mathematics and Statistics, University of Newcastle, Newcastle-upon-Tyne NE1 7RU, UK

Received 14 October 1994, in final form 16 January 1995

Abstract. A new Lax equation is introduced for the KP hierarchy which avoids the use of pseudo-differential operators, as used in the Sato approach. This Lax equation is closer to that used in the study of the dispersionless KP hierarchy, and is obtained by replacing the Poisson bracket with the Moyal bracket. The dispersionless limit, under which the Moyal bracket collapses to the Poisson bracket, is particularly simple.

1. Introduction

One of the simplest nonlinear equations that can be completely solved, albeit implicitly, is

$$4U_T - 12UU_X = 0$$

the solution to which can be obtained using the method of characteriztics. This equation can be described in two ways, either as the dispersionless KdV equation (i.e. the KdV equation without the dispersion U_{XXX} term) or as the simplest example of an equation of hydrodynamic type. This connection between dispersionless and hydrodynamic equations persists into the theory of $(2+1)$-dimensional systems. The dispersionless KP equation (dKP) (also known as the Khokhlov–Zabolotskaya equation)

$$(4U_t - 12UU_X)_X = 3U_{YY}$$

may be obtained from the KP equation itself:

$$(4u_t - 12uu_x - u_{xxx})_x = 3u_{yy}$$

via the scaling transformation

$$\begin{aligned} X &= \epsilon x \\ Y &= \epsilon y \\ T &= \epsilon t \\ U(X,Y,T) &= u(x,y,t) \end{aligned} \tag{1}$$

in the limit as $\epsilon \to 0$. The dispersionless KP equations (hereafter referred to as the dKP equations) may be reduced to the hydrodynamic-type equation

$$\frac{\partial w}{\partial T} = A(w)\frac{\partial w}{\partial X}$$

† E-mail: i.a.b.strachan@newcastle.ac.uk

where A is an $N \times N$ matrix and w is an N-component column vector (N, which characterizes this reduction, is an arbitrary integer). Such a reduction enables one to construct solutions to, and study the properties of, the dKP equation by using existing results on the theory of hydrodynamic type equations [1].

The dKP and KP equations are important examples of $(2+1)$-dimensional integrable systems, both having an associated Lax equation. For the KP equation (or, more generally, for the KP hierarchy) the Lax equation is [2]

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L] \tag{2}$$

where $\partial = \frac{\partial}{\partial x}$,

$$L = \partial + \sum_{n=2}^{\infty} v_n(x, t_2, t_3, \ldots)\partial^{-n+1}$$

and Λ_+ denotes the projection onto the differential operator part of the pseudo-differential operator Λ. The bracket $[A, B]$ is just the commutator of the differential operators, i.e. $[A, B] = A\,B - B\,A$.

The Lax equation for the dKP hierarchy is somewhat different, as it does not involve the use of pseudo-differential operators. The Lax equation is [1]

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{(\mathcal{L}^n)_+, \mathcal{L}\} \tag{3}$$

where

$$\mathcal{L} = \lambda + \sum_{n=2}^{\infty} u_n(x, t_2, t_3, \ldots)\lambda^{-n+1}$$

and Ω_+ denotes the projection onto positive (and zero) powers of λ in the Laurent series Ω. The bracket $\{f, g\}$ is just the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial \lambda}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial \lambda}\frac{\partial f}{\partial x}. \tag{4}$$

One interesting point to notice is that, although the scaling transformation takes one from the KP equation to the dKP equation, if one applies it to the Lax equation for the KP equation one does not obtain the Lax equation for the dKP equation, at least not in any naive way. This may be summarized as the failure of the following diagram to commute:

$$\begin{array}{ccc}
\text{KP hierarchy} & \longleftrightarrow & \text{Lax equation (2)} \\
\text{scaling (1)}\ \Big\downarrow & & \Big\downarrow \\
\text{dKP hierarchy} & \longleftrightarrow & \text{Lax equation (3).}
\end{array}$$

The aim of this paper is to introduce a new Lax equation for the KP hierarchy so that the above diagram does commute.

Another motivation comes from the programme to understand the structure of integrable systems from a geometric viewpoint—sometimes loosely stated as reformulating integrable systems as reductions of the self-dual Yang–Mills equations. The geometry of the dKP hierarchy is understood [3] but no similar description of the KP hierarchy has so far been found. The idea behind this paper is to reformulate the KP hierarchy in such a way as to be closer to the formulation of the dKP hierarchy, the hope being that this might facilitate the finding of this geometric description of the KP hierarchy. Thus procedures such as the scaling of variables—including λ—will be avoided as one might wish to retain the geometric properties they have for the dKP hierarchy. In fact one will obtain the KP equation in the form

$$(4u_t - 12uu_x - 4\kappa^2 u_{xxx})_x = 3u_{yy}$$

and so the dispersionless limit corresponds to $\kappa \to 0$. This then avoids the scaling transformation. This will be achieved by replacing the Poisson bracket in (3) by the Moyal bracket, and the dispersionless limit is the limit in which the Moyal bracket collapses to the Poisson bracket.

2. The Moyal bracket

The Moyal bracket [4] may be thought of as a deformation of the Poisson bracket by the introduction of higher-order derivative terms. It turns out that the Jacobi identity is highly restrictive as to the nature of these terms, and one is lead uniquely [5] to the Moyal bracket:

$$\{f, g\}_\kappa = \sum_{s=0}^{\infty} \frac{(-1)^s \kappa^{2s}}{(2s+1)!} \sum_{j=0}^{2s+1} (-1)^j \binom{2s+1}{j} (\partial_x^j \partial_\lambda^{2s+1-j} f)(\partial_x^{2s+1-j} \partial_\lambda^j g). \tag{5}$$

It has all the standard properties one would expect of such a bracket:

(i)	$\{f, g\}_\kappa = -\{g, f\}_\kappa$	antisymmetry
(ii)	$\{af + bg, h\}_\kappa = a\{f, h\}_\kappa + b\{g, h\}_\kappa$	linearity
(iii)	$\{f, \{g, h\}_\kappa\}_\kappa + \{g, \{h, f\}_\kappa\}_\kappa + \{h, \{f, g\}_\kappa\}_\kappa = 0$	Jacobi identity

(where a, b are independent of x and λ). Moreover, it has the important property that

$$\lim_{\kappa \to 0} \{f, g\}_\kappa = \{f, g\} \tag{6}$$

i.e. in the limit as $\kappa \to 0$ the bracket collapses to the Poisson bracket (4). It also has many other interesting properties [6], amongst which is the fact that it may be written in terms of an associative $\star$-product defined by

$$f \star g = \sum_{s=0}^{\infty} \frac{\kappa^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} (\partial_x^j \partial_\lambda^{s-j} f)(\partial_x^{s-j} \partial_\lambda^j g).$$

This has the property that

$$\lim_{\kappa \to 0} f \star g = fg \tag{7}$$

and with this the Moyal bracket takes the form:

$$\{f, g\}_\kappa = \frac{f \star g - g \star f}{2\kappa}.$$

The hierarchy to be considered here is obtained by replacing the Poisson bracket in (3) by the Moyal bracket:

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}_\kappa \tag{8}$$

where

$$\mathcal{B}_n = (\underbrace{\mathcal{L} \star \ldots \star \mathcal{L}}_{n\text{-terms}})_+$$

and $\mathcal{L}$ remains unchanged. This is equivalent to the zero-curvature relations

$$\frac{\partial \mathcal{B}_n}{\partial t_m} - \frac{\partial \mathcal{B}_m}{\partial t_n} + \{\mathcal{B}_n, \mathcal{B}_m\}_\kappa = 0. \tag{9}$$

Since the $\mathcal{B}_i$ are all polynomial in λ the Moyal bracket will automatically truncate after a finite number of terms, and so one obtains a well defined set of evolution equation for the independent variables. These equations differ from those of the dKP hierarchy by a finite number of κ-dependent terms. From (8) and (9) one may prove a number of general properties of the hierarchy. For example, using equations (6) and (7), as $\kappa \to 0$ the hierarchy reduces to the dKP hierarchy.

Example. With $n = 2$ and $n = 3$ equation (8) yields

$$\mathcal{B}_2 = \lambda^2 + 2u_2$$
$$\mathcal{B}_3 = \lambda^3 + 3\lambda u_2 + 3u_3$$

(κ-dependent terms only appear in $\mathcal{B}_n$ for $n > 3$), and one obtains from (9):

$$-3u_{2,t_2} + 6u_{3,x} = 0$$
$$2u_{2,t_3} - 6u_2u_{2,x} - 2\kappa^2 u_{2,xxx} - 3u_{3,t_2} = 0.$$

On eliminating u_3 one obtains a single equation for u_2:

$$(4u_{2,t_3} - 12u_2u_{2,x} - 4\kappa^2 u_{2,xxx})_x = 3u_{2,t_2t_2}$$

the KP equation itself. For this to agree with the KP equation obtained from (2) one has to set $\kappa^2 = \frac{1}{4}$. Further, as $\kappa \to 0$ one obtains the dKP directly without the need of the scaling transformation (1).

The remaining question is whether the hierarchies (2) and (8) are equivalent. The algebra of pseudo-differential operators is equivalent to the bracket

$$\{f, g\}'_\kappa = \sum_{n=0}^{\infty} \frac{\kappa^{2n+1}}{(2n+1)!} \left\{\partial_\lambda^{2n+1} f \partial_x^{2n+1} g - \partial_\lambda^{2n+1} f \partial_x^{2n+1} g\right\}$$

and this has many of the properties of the Moyal bracket. Thus the KP hierarchy may be reformulated in terms of this bracket instead of using pseudo-differential operators [7]. Thus the question of whether (2) and (8) are equivalent reduces to whether the brackets $\{f, g\}_\kappa$ and$\{f, g\}'_\kappa$ are equivalent. That they are follows from the fact that both are deformations of the Poisson bracket (4) and by the uniqueness of the Moyal bracket [5]. The difference being that $\{f, g\}'_\kappa$ corresponds to the standard ordering of operators and the Moyal bracket corresponds to the Weyl ordering of operators [8]. This Weyl ordering produces a slightly different—though equivalent—set of evolution equations, the functions v_n and u_n in equations (2) and (8) are related by simple changes of variable, the first few being (with $\kappa^2 = \frac{1}{4}$)

$$u_2 = v_2$$
$$u_3 = v_3 + \tfrac{1}{2}v_{2,x}$$
$$u_4 = v_4 + v_{3,x} + \tfrac{1}{4}v_{2,xx}.$$

Thus both these brackets may be used to formulate the KP hierarchy and both will give the dKP hierarchy in the limit $\kappa \to 0$. However, the Moyal bracket does seem to have certain advantages. It may be written as

$$\{f, g\}_\kappa = \sum_{n \text{ odd}} \{f, g\}_n \frac{\kappa^n}{n!}.$$

The coefficients $\{f, g\}_n$ are known as hyperJacobians and appear in topological field theory via the use of so-called null Lagrangians [9]. Moreover, they have very interesting geometrical properties. Just as the Possion bracket satifies the equation

$$\mathrm{d}u(x, y) \wedge \mathrm{d}v(x, y) = \{u, v\}\, \mathrm{d}x \wedge \mathrm{d}y$$

so the hyperJacobians satisfy

$$\mathrm{d}^n u(x, y) \wedge \mathrm{d}^n v(x, y) = \{u, v\}_n\, \mathrm{d}^n x \wedge \mathrm{d}^n y$$

where $\mathrm{d}^n x$ are an extension of the differential forms known as hyperforms [10]. Thus the Moyal bracket seems to have natural geometrical properties that may be of use in a future geometric description of the KP hierarchy.

A similar approach was studied in [11], however the formalism used there is slightly different and the equivalence of the two different types of bracket was not shown. Also, the $\kappa \to 0$ limit does not yield the dKP equation directly, not without a scaling transformation, the avoidance of which was one of the motivations of this section.

3. The reduction to the KdV hierarchy

The KdV hierarchy may be obtained by imposing the constraint $\mathcal{L}\star\mathcal{L} = \mathcal{B}_2$, and the evolution of u_2 is given by

$$\frac{\partial \mathcal{B}_2}{\partial t_{2n+1}} = \{\mathcal{B}_{2n+1}, \mathcal{B}_2\}_\kappa$$

all functions being independent of the even time variables. The first couple of equations are given below to show how the terms depend on the parameter κ:

$$\begin{aligned} u_{2,t_3} &= \kappa^2 u_{2,xxx} + 3u_2 u_{2,x} \\ u_{2,t_5} &= 10\kappa^4 u_{2,xxxxx} + 5\kappa^2 u_2 u_{2,xxx} + 10\kappa^2 u_{2,x} u_{2,xx} + \tfrac{15}{2} u_2^2 u_{2,x}. \end{aligned}$$

In the limit as $\kappa \to 0$ one obtains the dispersionless KdV hierarchy. Other (1+1)-dimensional hierarchies may be obtain by imposing the appropriate constraints, as in the standard Sato theory.

4. The geometry of the Moyal–KP hierarchy

A more geometrical way to describe the dKP hierarchy, equivalent to the Lax equation (3), is to introduce a 2-form†

$$\omega(\lambda) = \sum_{n=1}^{\infty} \mathrm{d}\widetilde{\mathcal{B}}_n \wedge \mathrm{d}t_n$$

where $\widetilde{\mathcal{B}}_n = (\mathcal{L}^n)_+$, i.e. the $\kappa \to 0$ limit of the $\mathcal{B}_n$. The dKP hierarchy then becomes the following conditions on the 2-form ω:

$$\begin{aligned} \omega(\lambda) \wedge \omega(\lambda) &= 0 \\ \mathrm{d}\omega(\lambda) &= 0. \end{aligned} \tag{10}$$

† In this section it will be notationally convenient to set $x = t_1$.

These equation imply, by Frobenius' theorem, the local existence of functions $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ such that $\omega = \mathrm{d}\mathcal{P} \wedge \mathrm{d}\mathcal{Q}$. In fact, one such pair of functions is given by

$$\begin{aligned}\mathcal{P}(\lambda) &= \mathcal{L} \\ \mathcal{Q}(\lambda) &= \sum_{n=1}^{\infty} nt_n \mathcal{L}^{n-1} \\ &\stackrel{\text{def}}{=} \mathcal{M}(\lambda)\end{aligned}$$

and hence:

Proposition [3]. The dispersionless KP hierarchy is governed by the exterior differential equation

$$\omega = \mathrm{d}\mathcal{L} \wedge \mathrm{d}\mathcal{M}$$

with

$$\{\mathcal{L}, \mathcal{M}\} = 1.$$

To discuss the geometry of the KP hierarchy it is first convenient to redefine the Moyal bracket

$$\{f, g\}_\kappa = f \star g - g \star f.$$

This amounts to rescaling the time variables, so now the limit $\kappa \to 0$ is singular. The basic definitions of $\mathcal{L}$ and $\mathcal{B}_n$ remains unchanged. In Sato theory the pseudo-differential operator

$$W = 1 + \sum_{n=1}^{\infty} w_i \partial^{-n}$$

plays a more fundamental role than the Lax operator L, and the evolution of W is governed by

$$\frac{\partial W}{\partial t_n} = B_n W - W \partial^n$$

where $L = W \partial W^{-1}$ and $B_n = (L^n)_+$. From these equations it is straightforward to derive the Lax equation and the zero-curvature relations. For the Moyal version of the KP hierarchy one may similarly define a function

$$\mathcal{W} = 1 + \sum_{n=1}^{\infty} w_n \lambda^n$$

governed by

$$\frac{\partial \mathcal{W}}{\partial t_n} = \mathcal{B}_n \star \mathcal{W} - \mathcal{W} \star \lambda^n.$$

The Lax functions is then $\mathcal{L} = \mathcal{W} \star \lambda \star \mathcal{W}^{-1}$ (where $\mathcal{W}^{-1}$ is defined uniquely by the relations $\mathcal{W} \star \mathcal{W}^{-1} = \mathcal{W}^{-1} \star \mathcal{W} = 1$), and this satisfies the Lax equation (8).

Recently, another pseudo-differential operator was introduced in [12],

$$M = W \left(\sum_{n=1}^{\infty} nt_n \partial^{n-1} \right) W^{-1}$$

this satisfying the equations

$$\frac{\partial M}{\partial t_n} = [B_n, M]$$
$$[L, M] = 1.$$

With this one may study the symmetries and other properties of the KP hierarchy in terms of the infinite dimensional Grassmannian manifold. Similarly, there is a Moyal version of this operator:

$$\mathcal{M} = \mathcal{W} \star \left(\sum_{n=1}^{\infty} n t_n \lambda^{n-1} \right) \star \mathcal{W}^{-1}.$$

This satisfies the relations

$$\frac{\partial \mathcal{M}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{M}\}_\kappa$$
$$\{\mathcal{L}, \mathcal{M}\}_\kappa = 1.$$

This suggests that one may characterize solutions of the Moyal KP hierarchy in terms of a Riemann–Hilbert problem in the Moyal loop group.

Another multidimensional integrable system that admits such a description is the anti-self-dual Einstein equation [13]. These describe a complex 4-metric with vanishing Ricci and anti-self-dual Weyl tensors. The metric may be written in terms of a single function Ω the Kähler potential, which is governed by the equation

$$\Omega_{,x\tilde{x}}\Omega_{,y\tilde{y}} - \Omega_{,x\tilde{y}}\Omega_{,y\tilde{x}} = 1$$

or, using the Poisson bracket (4) (with respect to $\tilde{x}$ and $\tilde{y}$ variables):

$$\{\Omega_{,x}, \Omega_{,y}\} = 1. \tag{11}$$

This equation (Plebanski's first heavenly equation [14]) may also be written in the form (10), with

$$\omega(\lambda) = \mathrm{d}x \wedge \mathrm{d}y + \lambda(\Omega_{x\tilde{x}}\,\mathrm{d}x \wedge \mathrm{d}\tilde{x} + \Omega_{x\tilde{y}}\,\mathrm{d}x \wedge \mathrm{d}\tilde{y} + \Omega_{y\tilde{x}}\,\mathrm{d}y \wedge \mathrm{d}\tilde{x} + \Omega_{y\tilde{y}}\,\mathrm{d}y \wedge \mathrm{d}\tilde{y}) + \lambda^2\,\mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}$$

and the additional constraint $\mathrm{d}\lambda = 0$. Once again one may show the existence of function $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ with $\omega = \mathrm{d}\mathcal{P} \wedge \mathrm{d}\mathcal{Q}$ and $\{\mathcal{P}(\lambda), \mathcal{Q}(\lambda)\} = 1$, connected by Riemann–Hilbert problems. A Moyal algebraic deformation of (11), obtained by replacing the Poisson bracket by the Moyal bracket was introduced in [15], and has been studied further by Takasaki [16] and Castro [17], the former showing that it may be described in terms of a Riemann–Hilbert problem in the corresponding Moyal loop group.

5. Comments

The Moyal bracket was first introduced in an attempt to reformulate quantum mechanics in terms of a distribution f on phase space. From the equation

$$\kappa \frac{\partial f}{\partial t} = \{f, H\}_\kappa$$

(and an auxiliary equation for f) one can derive a wavefunction

$$f = \int \bar{\psi}(x - y, t)\psi(x + y)\exp(\mathrm{i}py/\kappa)\,\mathrm{d}y$$

which satisfies Schrödinger equation [4]. Note that the use of commutator relations is avoided. The theory outlined in this paper is somewhat analogous; the use of a differential operator and commutator relations is replaced by the use of the Moyal bracket. In the approach of Jimbo and Miwa to the KP hierarchy one starts with a collection of free fermions, from which the τ-function is derived [18]. This suggests it may be possible to combine this approach with Moyal's idea of using phase-space distributions instead of commutator relations—there is certainly much similarity between the above formula of the wavefunction and Jimbo and Miwa's formula of the τ-function (theorem 2.1, [18]):

$$0 = \int \tau(x - \varepsilon(k^{-1}))\tau(x' + \varepsilon(k^{-1})) \exp(\xi(x - x'), k)\, dk$$

Perhaps the τ-function will have a simple description in terms of a phase-space description.

Using the ideas developed in this paper one may reformulate the entire theory of the KP hierarchy in terms of the Moyal bracket and $\star$-products, thus totally avoiding the use of pseudo-differential operators. One attraction of this approach is that it is closer in spirit to the formulation of other multidimensional integrable systems such as the anti-self-dual vacuum equations. For this equation the Riemann–Hilbert problem may be used to define an associated three-dimensional complex manifold known as twistor space. Such a twistorial description of the KP equation has been sought for many years, after the conjecture of Ward [19] that all classical integrable systems should admit such a description. The failure to find this has lead to the suggestion [20] that a more general version to twistor theory is needed to encompass systems such as the KP equation. The use of the Moyal bracket in the study of the KP hierarchy, as outlined in this paper, suggests that what might be required is some sort of κ-deformed twistor space which would make use of the Moyal bracket, rather than the Poisson bracket as used in conventional twistor theory. This, however, remains pure speculation.

Acknowledgments

Financial support was provided by the University of Newcastle, via the Wilfred Hall fellowship. I would like to thank Eduardo Ramos, Gerard Post and, in particular, David Fairlie for their comments on an earlier version of this paper.

References

[1] Kodama Y 19988 *Phys. Lett.* **129A** 223
Kodama Y and Gibbons J 1989 *Phys. Lett.* **135A** 167–70
[2] Ohta Y, Satuma J, Takahashi D and Tokihiro T 1988 *Prog. Theor. Phys. Supp.* **94** 210
[3] Takasaki K and Takebe T 1992 *Int. J. Mod. Phys.* A **7** Suppl. 1B 889
[4] Moyal J 1949 *Proc. Camb. Phil. Soc.* **45** 99
[5] Fletcher P 1990 *Phys. Lett.* **248B** 323
Arveson W 1983 *Commun. Math. Phys.* **89** 77
[6] Fairlie D B, Fletcher P and Zachos C K 1990 *J. Math. Phys.* **31** 1088
[7] Martinez-Moras F and Ramos E 1993 *Commun. Math. Phys.* **157** 573
Figueroa-O'Farrill J M, Mas J and Ramos E 1993 *Commun. Math. Phys.* **158** 17
[8] Fairlie D B and Nuyts J 1990 *Commun. Math. Phys.* **134** 413
Fairlie D B and Gelfand I M 1991 *Commun. Math. Phys.* **136** 487
[9] Fairlie D B 1991 *Proc. R. Soc. Edin.* A **119** 213
[10] Olver P J 1993 *Differential hyperforms I* University of Minesota Mathematics report 82–101; *Proc. R. Soc. Edin.* A **95** 317; 1987 *Invariant Theory (Lecture Notes in Mathematics 1278)* ed S S Koh (Berlin: Springer)
[11] Kupershmidt B A 1990 *Lett. Math. Phys.* **20** 19
[12] Grinevich P G and Orlov A Yu 1989 *Problems of Modern Quantum Field Theory* ed A A Belavin *et al* (Berlin: Springer)
[13] Penrose R 1976 *Gen. Rel. Grav.* **7** 31

[14] Plebanski J F *J. Math. Phys.* **16** 2395

[15] Strachan I A B 1992 *Phys. Lett.* **283B** 63

[16] Takasaki K 1994 *J. Geom. Phys.* **14** 111, 332

[17] Castro C 1995 Nonlinear W_∞ algebras from nonlinear integrable deformations of self dual gravity *Preprint* IAEC-4-94; 1995 A universal W_∞ algebra and quantization of integrable deformations of self-dual gravity *Preprint* IAEC-2-94

[18] Jimbo M and Miwa T 1983 *Publ. RIMS, Kyoto Univ.* **19** 943

[19] Ward R S 1985 *Phil. Trans. R. Soc.* A **315** 451; 1990 *Twistors in Mathematics and Physics* ed T N Bailey and R J Baston (Cambridge: Cambridge University Press)

[20] Mason L J 1995 *Twistor Theory (Lecture Notes in Pure and Applied Mathematics 169)* ed S Huggett (New York: Dekker)